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ON LAGRANGIAN SUBMANIFOLDS IN COMPLEX HYPERQUADRICS OBTAINED FROM ISOPARAMETRIC HYPERSURFACES

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INTRODUCTION

This article is based on my joint works with Associate Professor Hui Ma (Tsinghua University, Beijing).

A Lagrangian submanifold L is an n-dimensional submanifold immersed or embedded in a symplectic manifold (M^{2n}, ω) on which the symplectic form ω vanishes, and it is the most fundamental object in symplectic geometry. The study of Lagrangian submanifolds L in Kähler manifolds (M^n, ω, J, g) is a fruitful area in differential geometry of submanifolds. From various viewpoints of Riemannian geometry and symplectic geometry, there appear many interesting works on Lagrangian submanifolds in specific Kähler manifolds such as complex Euclidean spaces, complex projective spaces, complex space forms, Hermitian symmetric spaces, generalized flag manifolds and so on. Throughout this article, we treat compact immersed or embedded Lagrangian submanifolds without boundary.

In this article we shall explain our recent works on Lagrangian submanifolds in complex hyperquadrics $M^{2n} = Q_n(\mathbf{C})$ and their environs. The complex hyperquadric $M^{2n} = Q_n(\mathbf{C})$ is a compact Hermitian symmetric space of rank 2. There is a relationship between Lagrangian geoemtry in the complex hyperquadrics $Q_n(\mathbf{C})$ and hypersurface geometry in the standard unit sphere $S^{n+1}(1)$. Via the "Gauss maps "oriented hypersurfaces in $S^{n+1}(1)$ give Lagrangian submanifolds in $Q_n(\mathbf{C})$. Especially the Gauss images of oriented hypersurfaces with constant principal curvatures, so called "isoparametric hypersurfaces", in $S^{n+1}(1)$ constitute a nice class of compact minimal Lagrangian submanifolds embedded in $Q_n(\mathbf{C})$. By using the results of isoparametric

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hypersurface theory, we shall discuss (1) the properties of such Lagrangian submanifolds, (2) a classification of compact homogeneous Lagrangian submanifolds and (3) the Hamiltonian stability/instability of the Gauss images of homogeneous isoparametric hypersurfaces, in the complex hyperquadrics.

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1. Lagrangian submanifolds in symplectic manifolds and Hamiltonian deformations

1.1. Lagrangian submanifolds and Hamiltonian deformations.

Let (M^{2n}, ω) be a $2n$-dimensional symplectic manifold with symplectic form ω. By the definition a *Lagrangian immersion* $\varphi : L \longrightarrow M^{2n}$ is a smooth immersion of an n-dimensional (maximal dimensional) smooth manifold L into M satisfying the condition $\varphi^*\omega = 0$. If $\varphi : L \longrightarrow M^{2n}$ is a Lagrangian immersion, then by the non-degeneracy of ω the natural linear bundle map $\varphi^{-1}TM/\varphi_*TL \ni v \mapsto \alpha_v := \omega(v, \varphi_*(\cdot)) \in T^*L$ becomes a linear bundle isomorphism and thus we have a linear isomorphism $C^\infty(\varphi^{-1}TM/\varphi_*TL) \to \Omega^1(L)$.

Suppose that $\varphi_t : L \longrightarrow (M^{2n}, \omega)$ is a one-parameter smooth family of smooth immersions with $\varphi_0 = \varphi$. Let $V_t := \dfrac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^{-1}TM)$. Then we define

$$\{\varphi_t\} : \textit{Lagrangian deformation} \underset{\text{def}}{\Longleftrightarrow} \varphi_t \text{ is Lagr. imm. for each } t$$

$$\Longleftrightarrow \alpha_{V_t} \in Z^1(L) \ \text{ closed for each } t.$$

$$\{\varphi_t\} : \textit{Hamiltonian deformation} \underset{\text{def}}{\Longleftrightarrow} \alpha_{V_t} \in B^1(L) \ \text{ exact for each } t.$$

Hamiltonian deformations are Lagrangian deformations. The difference between Lagrangian deformations and Hamiltonian deformations is equal to $H^1(L; \mathbf{R}) \cong Z^1(L)/B^1(L)$. Particularly if $b_1(L) = 0$, then any Lagrangian deformation of L is a Hamiltonian deformation.

There is a characterization of a Hamiltonian deformation in terms of "isomonodromy deformation" as follows : Suppose that $\frac{1}{2\pi}[\omega] \in H^2(M, \mathbf{R})$ is an integral class. Then we know that there is a complex line bundle $\mathcal{L}$ over M with a $U(1)$-connection ∇ in $\mathcal{L}$ whose curvature coincides with $\sqrt{-1}\omega$. Let $\varphi_t : L \longrightarrow M$ be a Lagrangian deformation. For each t, we take the pull-back complex line bundle $\varphi_t^{-1}\mathcal{L}$ over L with the pull-back connection $\varphi_t^{-1}\nabla$ through φ_t and thus we have a family of flat connections $\{\varphi_t^{-1}\nabla\}$. Then

Lemma 1.1 (cf. [17], [25]). *$\{\varphi_t\}$ is a Hamiltonian deformation if and only if a family of flat connections $\{\varphi_t^{-1}\nabla\}$ has same holonomy homomorphism $\rho_t : \pi_1(L) \longrightarrow U(1)$.*

1.2. Lagrangian orbits and moment maps.

Proposition 1.1. *All Lagrangian orbits of Hamiltonian group action G on a symplectic manifold (M, ω) with moment map μ appears as components of the level set $\mu^{-1}(\alpha)$ for some $\alpha \in \mathfrak{z}(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ is the dual space of Lie algebra $\mathfrak{g}$ of G and*

$$\mathfrak{z}(\mathfrak{g}^*) := \{\alpha \in \mathfrak{g}^* \mid \mathrm{Ad}^*(a)\alpha = \alpha \textit{ for all } a \in G\}.$$

If M and G are compact and connected, then each Lagrangian orbit coincides with the level set $\mu^{-1}(\alpha)$ for some $\alpha \in \mathfrak{z}(\mathfrak{g}^*) \cong \mathfrak{c}(\mathfrak{g})$ the center of $\mathfrak{g}$.

2. Lagrangian submanifolds in Kähler manifolds

2.1. Hamiltonian minimality and Hamiltonian stability. Let (M, ω, J, g) be a Kähler manifold with complex structure J and Kähler metric g and $\varphi : L \longrightarrow M$ be a Lagrangian immersion. Let B denote the second fundamental form of L in (M, g).

$$\begin{array}{ll} H: & \text{mean curvature vector field of } \varphi \\ \updownarrow & \\ \alpha_H: & \textit{mean curvature form of } \varphi \end{array}$$

It is known ([11]) that the mean curvature form of a Lagrangian immersion always satisfies the identity

$$d\alpha_H = \varphi^* \rho_M$$

where ρ_M denotes the Ricci form of M defined by $\rho_M(X, Y) = \mathrm{Ric}^M(JX, Y)$ and Ric^M denotes the Ricci tensor field of (M, ω, J, g). Thus if M is Einstein-Kähler, then α_H is closed.

The notions of Hamiltonian minimality and Hamiltonian stability were introduced and investigated first by Y. G. Oh (1990) [22]. For the simplicity throughout this article we assume that L is compact without boundary.

φ : *Hamiltonian minimal* (or "*H-minimal*")

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^\forall\varphi_t : L \longrightarrow M$ Hamil. deform. with $\varphi_0 = \varphi$,

$$\frac{d}{dt}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0} = 0$$

$\iff \quad \delta\alpha_H = 0$

Moreover assume that φ is H-minimal. Then φ is called *Hamiltonian stable* if and only if for each Hamiltonian deformation $\{\varphi_t\}$ of $\varphi_0 = \varphi$,

$$\frac{d^2}{dt^2}\mathrm{Vol}\ (L, \varphi_t^* g)|_{t=0} \geq 0.$$

Lemma 2.1 (Hamiltonian Version of The Second Variational Formula [23]).

$$\frac{d^2}{dt^2}\mathrm{Vol}\ (L, \varphi_t^* g)|_{t=0}$$
$$= \int_L \left(\langle \Delta_L^1 \alpha, \alpha\rangle - \langle \overline{R}(\alpha), \alpha\rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_H, S\rangle + \langle \alpha_H, \alpha\rangle^2\right) dv$$

where Δ_L^1 denotes the Laplace operator of $(L, \varphi^* g)$ acting on the vector space $\Omega^1(L)$ of smooth 1-forms on L and

- $\alpha := \alpha_{\frac{\partial \varphi_t}{\partial t}|_{t=0}} \in B^1(L)$
- $\langle \overline{R}(\alpha), \alpha\rangle := \sum_{i,j=1}^{n} \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j), \quad \{e_i\}$: *o.n.b. of* T_pL
- $S(X, Y, Z) := \omega(B(X, Y), Z)$ *symmetric* 3*-tensor field on* L

Suppose that X is a holomorphic Killing vector field defined on M. Then the corresponding 1-form $\alpha_X := \omega(X, \cdot)$ on M is closed. If $H^1(M, \mathbf{R}) = \{0\}$, then $\alpha_X = \omega(X, \cdot)$ is exact, i.e. X is a Hamiltonian vector field on M. Hence we see that if M is simply connected, more generally $H^1(M, \mathbf{R}) = \{0\}$, then each holomorphic Killing vector field of M generates a volume-preserving Hamiltonian deformation of φ.

Definition 2.1. Such a Hamiltonian deformation of φ is called *trivial.*

Definition 2.2. Assume that φ is H-minimal. Then φ is called *strictly Hamiltonian stable* if the following two conditions are satisfied :

(i) φ is Hamiltonian stable.
(ii) The null space of the second variation on Hamiltonian deformations coincides with the vector subspace consisting of infinitesimal deformations induced by trivial Hamiltonian deformations of φ.

If L is strictly Hamiltonian stable, then L has local minimum volume under each Hamiltonian deformation.

Definition 2.3. Assume that (M, ω, J, g) is a Kähler manifold and G is an analytic subgroup of its automorphism group $\mathrm{Aut}(M, \omega, J, g)$. We call a Lagrangian orbit $L = G \cdot x \subset M$ of K a *homogeneous Lagrangian submanifold* of M.

Proposition 2.1. *Any compact homogeneous Lagrangian submanifold in a Kähler manifold is Hamiltonian minimal.*

Proof. Since α_H is an invariant 1-form on L, $\delta\alpha_H$ is a constant function on L. Hence by the divergence theorem we obtain $\delta\alpha_H = 0$. □

2.2. **First eigenvalue of minimal Lagrangian submanifolds in Einstein-Kähler manifolds.** By the Lagrangian version of the second variational formula, in the case of minimal Lagrangian submanifolds in Einstein-Kähler manifolds the Hamiltonian stability condition is simplified as follow :

Corollary 2.1 (B. Y. Chen - T. Nagano - P. F. Leung [9], Y. G. Oh [22]). *Assume M is an Einstein-Kähler manifold of Einstein constant κ and $\varphi : L \longrightarrow M$ is a minimal Lagrangian immersion of a compact smooth manifold L into M (i.e. $\alpha_H \equiv 0$). Then L is Hamiltonian stable if and only if*

$$\lambda_1 \geq \kappa ,$$

where λ_1 denotes the first (positive) eigenvalue of the Laplacian of L acting on $\Omega^0(L) = C^\infty(L)$.

Theorem 2.1 ([26], [27], [4]). *Assume that M is a compact homogeneous Einstein-Kähler manifold with Einstein constant $\kappa > 0$. Let $L \hookrightarrow M$ be a compact minimal Lagrangian submanifold immersed in M. Then*

$$\lambda_1 \leq \kappa .$$

Question. What compact minimal Lagrangian submanifolds in a compact homogeneous Einstein-Kähler manifold M with $\kappa > 0$ attain the equality of the inequality $\lambda_1 \leq \kappa$?

$$\lambda_1 = \kappa \quad \Longleftrightarrow \quad L \text{ is Hamiltonian stable.}$$

2.3. **Examples of Hamiltonian stable Lagrangian submanifolds.**

Question. What compact Lagrangian submanifolds in a Kähler manifold is a Hamiltonian stable H-minimal Lagrangian submanifold ?

Example 2.1. (1) $S^1 \subset \mathbf{R}^2 \cong \mathbf{C}$, $S^1 \subset S^2 \cong \mathbf{C}P^1$, $S^1 \subset H^2 \cong \mathbf{C}H^1$, circles

(2) $L \hookrightarrow M = \mathbf{C}P^n$ embedded cpt. min. Lagr. submfd.
$L =$

- $\mathbf{R}P^n$ (Y. G. Oh [22]), $S = 0$.
- $SU(p)/SO(p)\cdot\mathbb{Z}_p$, $SU(p)/\mathbb{Z}_p$, $SU(2p)/Sp(p)\cdot\mathbb{Z}_{2p}$, $E_6/F_4\cdot\mathbb{Z}_3$
(Amarzaya-Ohnita [4]), $S \neq 0, \nabla S = 0$
- $\rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbf{C}P^3$ (L. Bedulli-A. Gori [6], Ohnita [24]). $\nabla S \neq 0$

$\Longrightarrow$ L is strictly Hamil. stable

(3) M : cpt. irred. Herm. sym. sp. of rank ≥ 2
L : cpt. totally geodesic Lagrangian submanifold embedded in M.
Then

$$\underset{\substack{\text{tot. geod.}\\ \text{Lagr. submfd.}}}{(L, M)} = \begin{cases} (Q_{p,q}(\mathbf{R}) = (S^{p-1} \times S^{q-1})/\mathbf{Z}_2, \\ \quad Q_{p+q-2}(\mathbf{C}))(q - p \geq 3) \\ (U(2p)/Sp(p), SO(4p)/U(p))(p \geq 3), \\ (T \cdot E_6/F_4, E_7/T \cdot E_6). \end{cases}$$

$\Longleftrightarrow$ L is NOT Hamil. stable.
(Masaru Takeuchi [32], Y. G. Oh [22], Amarzaya-Ohnita [4], cf. [17])

Theorem 2.2 (Amarzaya-Ohnita [2], [5]). *Let $L^n \hookrightarrow \widetilde{M}(c)$ be a compact embedded Lagrangian submanifold with $\nabla S = 0$ in a simply connected complete complex space form $\widetilde{M}(c)(= \mathbf{C}^n, \mathbf{C}P^n, \mathbf{C}H^n)$. Then L is Hamiltonian stable.*

Problem. Let $L \hookrightarrow \mathbf{C}P^n$ be a compact minimal Lagrangian submanifold *embedded* in a complex projective space. Is it true that $\lambda_1 = \kappa$? that is, L is Hamiltonian stable ? (At present I do not know any counter example yet.)

3. Lagrangian Submanifolds in Complex Hyperquadrics

3.1. Complex hyperquadrics and real Grassmannian manifolds of oriented 2-planes. The *complex hyperquadric*

$$Q_n(\mathbf{C}) \cong \widetilde{Gr_2}(\mathbf{R}^{n+2}) \cong SO(n+2)/SO(2) \times SO(n)$$

is a compact Hermitian symmetric space of rank 2, where

$$Q_n(\mathbf{C}) := \{[\mathbf{z}] \in \mathbf{C}P^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\},$$

$$\widetilde{Gr}_2(\mathbf{R}^{n+2}) := \{W \mid \text{oriented 2-dimensional vector subspace of } \mathbf{R}^{n+2}\}.$$

The identification between $Q_n(\mathbf{C})$ and $\widetilde{Gr}_2(\mathbf{R}^{n+2})$ is given by

$$\mathbf{C}P^{n+1} \supset Q_n(\mathbf{C}) \ni [\mathbf{a}+\sqrt{-1}\mathbf{b}] \longleftrightarrow W = \mathbf{a}\wedge\mathbf{b} \in \widetilde{Gr}_2(\mathbf{R}^{n+2}) \subset \bigwedge^2 \mathbf{R}^{n+2}$$

Here $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis of W compatible with its orientation.

3.2. **Lagrangian submanifolds in complex hyperquadrics and hypersurfaces in spheres.** Let $N^n \hookrightarrow S^{n+1}(1) \subset \mathbf{R}^{n+2}$ be an oriented hypersurface immersed or embedded in the $(n+1)$-dimensional unit standard sphere.

Let $\mathbf{x}$ and $\mathbf{n}$ denote the position vector of points of N^n and the unit normal vector field of N^n in $S^{n+1}(1)$, respectively.

The "Gauss map"

$$\mathcal{G} : N^n \ni p \longmapsto [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p) \wedge \mathbf{n}(p) \in Q_n(\mathbf{C})$$

is a Lagrangian immersion.

Proposition 3.1 ([17]). *Any deformation of oriented hypersurface N^n in $S^{n+1}(1)$ gives a Hamiltonian deformation of $\mathcal{G}$ in $Q_n(\mathbf{C})$. Conversely, any small Hamiltonian deformation of $\mathcal{G}$ in $Q_n(\mathbf{C})$ is obtained from a deformation of oriented hypersurface N^n in $S^{n+1}(1)$.*

Remark. $(2n+1)$-dimensional real Stiefel manifold

$$V_2(\mathbf{R}^{n+2}) := \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbf{R}^{n+2} \text{ orthonormal }\} \cong SO(n+2)/SO(n)$$

the standard Einstein-Sasakian manifold over $Q_n(\mathbf{C})$.

The natural projections

$$p_1 : V_2(\mathbf{R}^{n+2}) \ni (\mathbf{a}, \mathbf{b}) \longmapsto \mathbf{a} \in S^{n+1}(1),$$

$$p_2 : V_2(\mathbf{R}^{n+2}) \ni (\mathbf{a}, \mathbf{b}) \longmapsto \mathbf{a} \wedge \mathbf{b} \in Q_n(\mathbf{C}).$$

$$\begin{tikzcd}
\tilde{N}^n \arrow[r, "\psi", "\text{Legend.}"'] \arrow[d, "\cong"'] & V_2(\mathbf{R}^{n+2}) \arrow[d, "p_1"', "S^n"] \arrow[r, equal] & V_2(\mathbf{R}^{n+2}) \arrow[d, "p_2"', "S^1"] \\
N^n \arrow[r, "\text{ori.hypsurf.}"'] & S^{n+1}(1) & Q_n(\mathbf{C}) \supset p_2(\psi(N^n)) = \mathcal{G}(N^n) \\
 & & \text{Lagr.}
\end{tikzcd}$$

Here the Legendrian life $\tilde{N}^n$ of $N^n \hookrightarrow S^{n+1}(1)$ to $V_2(\mathbf{R}^{n+2})$ is defined by $N^n \ni p \longmapsto (\mathbf{x}(p), \mathbf{n}(p)) \in V_2(\mathbf{R}^{n+2})$.

3.3. **The mean curvature form formula.** Let $g^{std}_{Q_n(\mathbf{C})}$ be the standard Kähler metric of $Q_n(\mathbf{C})$ induced from the standard inner product of $\mathbf{R}^{n+2}$. Note that the Einstein constant of $g^{std}_{Q_n(\mathbf{C})}$ is equal to n. Let κ_i $(i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$. Choose an orthonormal frame $\{e_i\}$ on $N^n \subset \mathbf{R}^{n+1}$ such that the second fundamental form h of N^n in $S^{n+1}(1)$ with respect to $\mathbf{n}$ is diagonalized as $h(e_i, e_j) = \kappa_i \delta_{ij}$ and let $\{\theta^i\}$ be its dual coframe. Then the induced metric $\mathcal{G}^* g^{std}_{Q_n(\mathbf{C})}$ on N^n by the Gauss map $\mathcal{G}$ is given as

$$\mathcal{G}^* g^{std}_{Q_n(\mathbf{C})} = \sum_{i=1}^{n} (1 + \kappa_i^2) \theta^i \otimes \theta^i.$$

Let H denote the mean curvature vector field of $\mathcal{G}$. Then the mean curvature form of the Gauss map $\mathcal{G}$ is expressed in terms of the principal curvatures as follows :

Lemma 3.1 (B. Palmer [30]).

$$\alpha_H = d \left(\operatorname{Im} \left(\log \prod_{i=1}^{n} (1 + \sqrt{-1}\kappa_i) \right) \right).$$

In case $n = 2$, if $N^2 \subset S^3(1)$ is a minimal surface, then the Gauss map $\mathcal{G} : N^2 \longrightarrow \widetilde{Gr}_2(\mathbf{R}^4) \cong Q_2(\mathbf{C}) \cong S^2 \times S^2$ is a minimal Lagrangian immersion. See also Castro-Urbano [8]. More generally, if $N^n \subset S^{n+1}(1)$ is an oriented minimal hypersurface in $S^{n+1}(1)$ which is an *austere* submanifold of $S^{n+1}(1)$ (Harvey-Lawson [14]), then the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is a minimal Lagrangian immersion.

3.4. **The Gauss maps of isoparametric hypersurfaces in $S^{n+1}(1)$.** Assume that $N^n \hookrightarrow S^{n+1}(1) \subset \mathbf{R}^{n+2}$ is a compact oriented hypersurface embedded in the standard sphere with constant principal curvatures, so called "*isoparametric hypersurface*". Here g denotes the number of distinct principal curvatures of N^n in $S^{n+1}(1)$ and $m_1, m_2, \cdots, m_g$ denote the multiplicities of the principal curvatures. Then the image of the Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is a compact minimal Lagrangian submanifold embedded in $Q_n(\mathbf{C})$ and the Gauss map gives a covering map $N^n \underset{\mathbb{Z}_g}{\longrightarrow} L^n = \mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g \hookrightarrow Q_n(\mathbf{C})$ with Deck transformation group $\mathbf{Z}_g$.

By the famous theorems of H. F. Münzner [20], [21], we know that m_i $(i = 1, \cdots, g)$ satisfy $m_i = m_{i+2}$ for each i, i.e., $m_1 = m_3 = \cdots$, $m_2 = m_4 = \cdots$, and g must be 1, 2, 3, 4 or 6. We may assume that $m_1 \leq m_2$.

If a hypersurface N^n in $S^{n+1}(1)$ is obtained as an orbit of a compact connected subgroup G of $SO(n+2)$, then N^n is called *homogeneous*. Obviously a homogeneous hypersurface in $S^{n+1}(1)$ is an isoparametric hypersurface in $S^{n+1}(1)$.

Proposition 3.2 ([17]). *N^n is homogeneous if and only if $\mathcal{G}(N^n)$ is homogeneous.*

In [17] we classified all compact homogeneous Lagangian submanifolds in complex hyperquadrics $Q_n(\mathbf{C})$ by using the theory of homogeneous isoparametric hypersurfaces. We shall mention it in the next subsection.

By W.-Y. Hsiang and J. B. Lawson, Jr. [15] and Ryoichi Takagi and Tsunero Takahashi [31] all homogeneous isoparametric hypersurfaces in the spheres are obtained as principal orbits of the linear isotropy representations of Riemannian symmetric spaces (U, K) of rank 2.

g	(U,K)	$\dim N$	m_1, m_2	$N \cong K/K_0$
1	$(S^1 \times SO(n+2), SO(n+1))$ $(n \geq 1)$ $[\mathbf{R} \oplus \mathrm{A}_1]$	n	n	S^n
2	$(SO(p+2) \times SO(n+2-p),$ $SO(p+1) \times SO(n+1-p))$ $(1 \leq p \leq n-1)$ $[\mathrm{A}_1 \oplus \mathrm{A}_1]$	n	$p, n-p$	$S^p \times S^{n-p}$
3	$(SU(3), SO(3))$ $[\mathrm{A}_2]$	3	$1, 1$	$\frac{SO(3)}{\mathbf{Z}_2+\mathbf{Z}_2}$
3	$(SU(3) \times SU(3), SU(3))$ $[\mathrm{A}_2]$	6	$2, 2$	$\frac{SU(3)}{T^2}$
3	$(SU(6), Sp(3))$ $[\mathrm{A}_2]$	12	$4, 4$	$\frac{Sp(3)}{Sp(1)^3}$
3	(E_6, F_4) $[\mathrm{A}_2]$	24	$8, 8$	$\frac{F_4}{Spin(8)}$
4	$(SO(5) \times SO(5), SO(5))$ $[\mathrm{B}_2]$	8	$2, 2$	$\frac{SO(5)}{T^2}$
4	$(SU(m+2), S(U(m) \times U(2)))$ $(m \geq 2)$ $[\mathrm{BC}_2](m \geq 3), [\mathrm{B}_2](m=2)$	$4m-2$	$2,$ $2m-3$	$\frac{S(U(m)\times U(2))}{SU(m-2)\times T^2}$
4	$(SO(m+2), SO(m) \times SO(2))$ $(m \geq 3)$ $[\mathrm{B}_2]$	$2m-2$	$1,$ $m-2$	$\frac{SO(m)\times SO(2)}{SO(m-2)\times \mathbf{Z}_2}$
4	$(Sp(m+2), Sp(m) \times Sp(2))$ $(m \geq 2)$ $[\mathrm{BC}_2](m \geq 3), [\mathrm{B}_2](m=2)$	$8m-2$	$4,$ $4m-5$	$\frac{Sp(m)\times Sp(2)}{Sp(m-2)\times Sp(1)^2}$
4	$(SO(10), U(5))$ $[\mathrm{BC}_2]$	18	$4, 5$	$\frac{U(5)}{SU(2)\times SU(2)\times T^1}$
4	$(E_6, Spin(10) \cdot T)$ $[\mathrm{BC}_2]$	30	$6, 9$	$\frac{Spin(10)\cdot T}{SU(4)\cdot T}$
6	$(G_2 \times G_2, G_2)$ $[\mathrm{G}_2]$	12	$2, 2$	$\frac{G_2}{T^2}$
6	$(G_2, SO(4))$ $[\mathrm{G}_2]$	6	$1, 1$	$\frac{SO(4)}{\mathbf{Z}_2+\mathbf{Z}_2}$

In the case of $g = 4$, the Clifford algebra construction of non-homogeneous isoparametric hypersurfaces in the sphere were discovered first by Hideki Ozeki and Masaru Takeuchi [28], [29] and generalized by D. Ferus, H. Karcher and H. F. Münzner [13] (so called "isoparametric hypersurfaces of OT-FKM type"). Recently T. Cecil, Q.-S. Chi and G. Jensen [10] and S. Immervoll [16] showed that isoparametric hypersurfaces in the sphere with $g = 4$ except for the cases of $(m_1, m_2) = (3, 4), (4, 5), (6, 9), (7, 8)$ are either homogeneous or of type OT-FKM type.

3.5. **Classification of compact homogeneous Lagrangian submanifolds in complex hyperquadrics.** Suppose that $G \subset SO(n+2)$ is a compact connected Lie subgroup and $L = G \cdot [W] \subset Q_n(\mathbf{C})$ is a Lagrangian orbit of G through a point $[W] \in Q_n(\mathbf{C})$

W is an oriented 2-dimensional vector subspace of $\mathbf{R}^{n+2}$ and we denote a unit circle of W by

$$S^1(W) := \{v \in W \mid \|v\| = 1\}.$$

Then we can show that there is a finite subset $w_1, \cdots, w_d$ of $S^1(W)$ such that for each $w \in S^1(W) \setminus \{w_1, \cdots, w_d\}$ the orbit $G \cdot w$ of G through w on $S^{n+1}(1) \subset \mathbf{R}^{n+1}$ is a compact homogeneous hypersurface in $S^{n+1}(1)$ ([17]). We set $N^n := G \cdot w$.

By the Hsiang-Lawson's theorem, There is a compact Riemannian symmetric pair (U, K) of rank 2 such that

$$N^n = \mathrm{Ad}_{\mathfrak{p}}(K)v \subset S^{n+1}(1) \subset \mathbf{R}^{n+2} = \mathfrak{p} \quad \text{for some } v \in S^{n+1}(1),$$

where $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ is the canonical decomposition of the symmetric pair (U, K). Here we may assume that $\mathrm{Ad}_{\mathfrak{p}}(K) \subset SO(n+2)$ is a maximal compact subgroup of $SO(n+2)$ containing G which is orbit-equivalent to the action of G on $S^{n+1}(1)$.

Then we obtain

Theorem 3.1 (Hui Ma-Y. Ohnita [17]). *There exists a compact homogeneous isoparametric hypersurface $N^n \subset S^{n+1}(1) \subset \mathbf{R}^{n+2}$ such that*

(i) *$L = \mathcal{G}(N)$ and L is a compact minimal Lagrangian submanifold, or*

(ii) *L is contained in a Lagrangian deformation of $\mathcal{G}(N)$ consisting of compact homogeneous Lagrangian submanifolds.*

The second case (ii) happens only when (U, K) is one of

(1) $(S^1 \times SO(3), SO(2))$,
(2) $(SO(3) \times SO(3), SO(2) \times SO(2))$,
(3) $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ $(n \geq 3)$,

(4) $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2, m \geq 3)$.

In the first two cases, it is elementary and well-known to describe all Lagrangian orbits of the natural actions of $K = SO(2)$ on $Q_1(\mathbf{C}) \cong S^2$ and $K = SO(2) \times SO(2)$ on $Q_2(\mathbf{C}) \cong S^2 \times S^2$. Also in the last two cases there exist one-parameter families of Lagrangian K-orbits in $Q_n(\mathbf{C})$ and each family contains Lagrangian submanifolds which can NOT be obtained as the Gauss image of any homogeneous isoparametric hypersurface in a sphere. The fourth one is a new family of Lagrangian orbits.

(1) If (U, K) is $(S^1 \times SO(3), SO(2))$, then L is a small or great circle in $Q_1(\mathbf{C}) \cong S^2$.

(2) If (U, K) is $(SO(3) \times SO(3), SO(2) \times SO(2))$, then L is a product of small or great circles of S^2 in $Q_2(\mathbf{C}) \cong S^2 \times S^2$.

(3) If (U, K) is $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ $(n \geq 2)$, then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbf{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic K-orbits satisfying

(a) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a totally geodesic Lagrangian submanifold in $Q_n(\mathbf{C})$.

(b) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,

$$K \cdot [W_\lambda] \cong (S^1 \times S^{n-1})/\mathbf{Z}_2 \cong Q_{2,n}(\mathbf{R})$$

is an H-minimal Lagrangian submanifold in $Q_n(\mathbf{C})$ with $\nabla S = 0$ and thus $\nabla \alpha_H = 0$.

(c) $K \cdot [W_{\pm\sqrt{-1}}]$ are isotropic submanifolds in $Q_n(\mathbf{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = 0$ (points !).

(4) If (U, K) is $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2)$, then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbf{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagrangian or isotropic orbits satisfying

(a) $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a minimal (NOT totally geodesic) Lagrangian submanifold in $Q_n(\mathbf{C})$.

(b) For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,

$$K \cdot [W_\lambda] \cong (SO(2) \times SO(m))/(\mathbf{Z}_2 \times \mathbf{Z}_4 \times SO(m-2))$$

is an H-minimal Lagrangian submanifold in $Q_n(\mathbf{C})$ with $\nabla S \neq 0$ and $\nabla \alpha_H = 0$.

(c) $K \cdot [W_{\pm\sqrt{-1}}] \cong SO(m)/S(O(1) \times O(m-1)) \cong \mathbf{R}P^{m-1}$ are isotropic submanifolds in $Q_n(\mathbf{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = m-1$.

3.6. **Hamiltonian stability of the Gauss images of homogeneous isoparametric hypersurfaces in $S^{n+1}(1)$.** Suppose that N^n is a compact isoparametric hypersurface embedded in $S^{n+1}(1)$. Palmer [30] showed that its Gauss map $\mathcal{G} : N^n \longrightarrow Q_n(\mathbf{C})$ is Hamiltonian stable if and only if $N^n = S^n \subset S^{n+1}(1)$ $(g = 1)$.

Question. Hamiltonian stability of its Gauss image $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$?

$g = 1 :$ $N^n = S^n$ is a great or small sphere and $\mathcal{G}(N^n) \cong S^n$ is strictly Hamiltonian stable. More strongly, it is stable as a minimal submanifold and homologically volume-minimizing because it is a calibrated submanifold.

$g = 2 :$ $N^n = S^{m_1} \times S^{m_2}$ the Clifford hypersurfaces $(n = m_1 + m_2, 1 \leq m_1 \leq m_2)$ and $\mathcal{G}(N^n) = Q_{m_1+1,m_2+1}(\mathbf{R}) \subset Q_n(\mathbf{C})$. Then $m_2 - m_1 \geq 3$ if and only if $\mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is NOT Hamiltonian stable. In case $m_2 - m_1 \geq 3$, the spherical harmonics of degree 2 on the sphere $S^{m_1} \subset \mathbf{R}^{m_1+1}$ of smaller dimension give volume-decreasing Hamiltonian deformations of $\mathcal{G}(N^n)$. If $m_2 - m_1 = 2$, then it is Hamiltonian stable but not strictly Hamiltonian stable. If $m_2 - m_1 = 0$ or 1, then it is strictly Hamiltonian stable.

$g = 3 :$ All isoparametric hypersurfaces in the sphere with $g = 3$ were classified by E. Cartan and they all are homogeneous, so called "Cartan hypersurfaces".

Theorem 3.2 (Hui Ma-Ohnita [17]). *If $g = 3$, then $L = \mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is strictly Hamiltonian stable.*

Remark. In case $g = 3$, each induced metric from $Q_n(\mathbf{C})$ is a normal homogeneous metric. It never holds in cases $g = 4, 6$

$g = 6 :$ Only homogeneous examples are known now. If $g = 6$ and $m_1 = m_2 = 1$, then it is homogeneous (Dorfmeister-Neher [12], Reiko Miyaoka [18]).

Theorem 3.3 (Hui Ma-Ohnita). *If $g = 6$ and N^n is homogeneous, then $L = \mathcal{G}(N^n) \subset Q_n(\mathbf{C})$ is strictly Hamiltonian stable.*

$g = 4 :$ More recently, in the case when $g = 4$ and N^n is homogeneous, we obtain

Theorem 3.4 (Hui Ma-Ohnita). (1)

$$\mathcal{G}(N^n) = SO(5)/T^2 \cdot \mathbf{Z}_4$$

is strictly Hamiltonian stable.

(2)

$$\mathcal{G}(N^n) = (SO(2) \times SO(m))/(\mathbf{Z}_2 \times SO(m-2)) \cdot \mathbf{Z}_4 \, (m \geq 3)$$

is NOT Hamiltonian stable if and only if $m \geq 6$, *i.e.* $m_2 - m_1 = (m-2) - 1 \geq 3$. *If* $m_2 - m_1 = (m-2) - 1 = 2$, *i.e.* $m = 5$, *then it is Hamiltonian stable but not strictly Hamiltonian stable. If* $m_2 - m_1 = (m-2) - 1 = 0$ *or* 1, *i.e.* $m = 3$ *or* 4, *then it is strictly Hamiltonian stable.*

Our study on the Hamiltonian stability of their Gauss images in the homogeneous cases of $g = 4$ is still in progress and we shall report further results in this case on the forthcoming opportunity.

Problem. Investigate the Hamiltonian stability of the Gauss images of compact non-homogeneous isoparametric hypersurface embedded in the sphere with $g = 4$. For every compact isoparametric hypersurface embedded in the sphere, is it true that its Gauss image is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$?

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